4.2 REDUCTION OF ORDER

IF y_1 IS A SOL'N OF $y''+py'+qy=0$

LET $y_2 = vy_1$ (WILL TRANSFORM INTO A 1^{ST} ORDER SEPARABLE ODE)

CHECK THAT $x=e^t$ IS A SOL'N OF $tx''-(t+1)x'+x=0$

AND FIND THE GENERAL SOL'N

$$te^t-(t+1)e^t+e^t$$
$$=e^t(t-t-1+1)=0 \checkmark$$

SO $x = c_1e^t + c_2x_2$ (FIND x_2)

$$x_2 = vx_1 = ve^t$$
$$x_2' = v'e^t+ve^t$$
$$x_2'' = v''e^t+v'e^t+v'e^t+ve^t$$
$$= v''e^t+2v'e^t+ve^t$$

$$tx_2''-(t+1)x_2'+x_2$$
$$= e^t\left(tv''+2tv'+\cancel{tv}-(t+1)v'-\cancel{(t+1)v}+\cancel{v}\right)$$
$$= \underbrace{e^t}_{\neq 0}\left(tv''+(t-1)v'\right)=0$$

MANDATORY CHECKPOINT: NO V WITHOUT '

$$tv''+(t-1)v'=0$$

LET $u=v' \longrightarrow u'=v''$

$tu' + (t-1)u = 0$

MANDATORY CHECKPOINT:

LINEAR

$u' + \frac{t-1}{t}u = 0$

$\mu = e^{\int \frac{t-1}{t}dt} \ldots$

OR

$t\frac{du}{dt} + (t-1)u = 0$

$t\frac{du}{dt} = (1-t)u$

$t\,du = (1-t)u\,dt$

$\int \frac{1}{u}du = \int \frac{1-t}{t}dt$

MANDATORY CHECKPOINT:

SEPARABLE

$\ln|u| = \ln|t| - t + \cancel{C}$

$v' = u = \cancel{C}te^{-t}$

$$\begin{array}{ll} t & +\; e^{-t} \\ 1 & -\; e^{-t} \\ 0 & \;\; e^{-t} \end{array}$$

$v = -te^{-t} - e^{-t}$

$v = -(t+1)e^{-t}$

$x_2 = -(t+1)e^{-t} \cdot e^{t}$

$x_2 = -(t+1)$ $\qquad x_1 = e^{t}$

$x = c_1e^{t} + c_2(t+1)$

WARNING ABOUT $+, -$

$\ln|u| = -\ln|t| + t$

$v' = u = t^{-1}e^{t}$

$v = \int t^{-1}e^{t}dt$ (CAN'T BE DONE IN ELEM FUNC'S)

$\ln|u| = \ln|t| + t$

$v' = u = te^{t}$

$v = \int te^{t}dt$ (IBP)

4.7 CAUCHY-EULER EQUATIONS

↳ HAS FORM $Ax^2y'' + Bxy' + Cy = 0$ {OR $g(x)$}

eg. $x^2y'' + 7xy' - 7y = 0$ → E+U

$$\underbrace{y''}_{} + \underbrace{\frac{7}{x}}_{p(x)}y' - \underbrace{\frac{7}{x^2}}_{q(x)}y = \underbrace{0}_{g(x)}$$

EACH SUCCESSIVE DERIVATIVE LOSES AN x
(IE. POWER OF x DROPS BY 1)

p, q, g ARE CONT. ON $(-\infty, 0)$ OR $(0, \infty)$

SO THERE IS A UNIQUE SOL'N ON $(-\infty, 0)$ OR $(0, \infty)$

GUESS $y = x^r$

$y' = rx^{r-1}$

$y'' = r(r-1)x^{r-2}$

$x^2y'' + 7xy' - 7y$

$= r(r-1)x^r + 7r\,x^r - 7x^r = 0$ ← FUNCTION

$$\underbrace{(r(r-1) + 7r - 7)}_{=0}\underbrace{x^r}_{\neq 0 \text{ FUNCTION}} = 0$$

$r(r-1) + 7(r-1) = 0$

$(r+7)(r-1) = 0 \longrightarrow r = -7, 1$

YOU: CHECK
$y = x^{-7}$ AND $y = x^1 = x$
ARE BOTH SOL'NS

CLEARLY, NOT CONSTANT MULTIPLES OF EACH OTHER

SO, $y = C_1x^{-7} + C_2x$

$$Ax^2y'' + Bxy' + Cy = 0$$

GUESS $y = x^r$

$y' = rx^{r-1}$

$y'' = r(r-1)x^{r-2}$

$$Ax^2y'' + Bxy' + Cy = \left(Ar(r-1) + Br + C\right)x^r = 0$$

$$Ar(r-1) + Br + C = 0$$

$$Ar^2 - Ar + Br + C = 0$$

$$Ar^2 + (B-A)r + C = 0$$ ← INDICIAL POLYNOMIAL

IF $r = r_1, r_2$

$y = c_1 x^{r_1} + c_2 x^{r_2}$

BUT WHAT IF

$r_1 = r_2$ OR

$r_1 = a + bi$

AND $r_2 = a - bi$

?

SANITY CHECK: $x^2y'' + 7xy' - 7y = 0 \rightarrow \boxed{r^2 + 6r - 7 = 0}$

$A = 1,\ B = 7,\ C = -7$

$(r+7)(r-1) = 0$

$\boxed{r = -7, 1}$

$\boxed{y = c_1 x^{-7} + c_2 x}$

$$Ax^2y'' + Bxy' + Cy = 0$$

LET $x = e^t \longrightarrow t = \ln x$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

$$= \frac{dy}{dt}\frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$y'' = \frac{d^2y}{dx^2}$$

?